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The effect of initial stress and magnetic field on wave propagation in human dry bones

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Abstract

The aim of the present paper is to study the influence of initial stress and magnetic field on the propagation of harmonic waves in a human long dry bone as transversely isotropic material, subject to the boundary conditions that the outer and inner surfaces are traction free. The equations of elastodynamics are solved in terms of displacements. The natural frequency of plane vibrations in the case of harmonic vibrations has been obtained. The frequencies and phase velocity are calculated numerically, the effects of initial stress and magnetic field are discussed. Comparisons are made with the result in the absence of initial stress and magnetic field.

MSC: 74B05

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1 Introduction

The investigation of wave propagation over a continuous medium has very important application in the fields of engineering, medicine and in bioengineering. Application of the poroelastic materials in medicinal fields such as cardiovascular, dental and orthopedics is well known. The dry bone is piezoelectric in the classical sense [1, 2], *i.e.*, mechanical stress results in electric polarization (the indirect effect); and an applied electric field causes strain (the converse effect). Since that time, many others have confirmed the capacity of bones to produce piezoelectric potentials [3]. Electrical properties of bone are relevant not only as a hypothesized feedback mechanism for bone adaptation and remodeling, but also in the context of external electrical stimulation of bone in order to aid its healing and repair [4]. In orthopedics, the propagation of wave over bone is used in monitoring the rate of fracture healing. There are two types of osseous tissue such as trabecular or cancellous and cortical or compact bone, which are of different materials with respect to their mechanical behavior. In macroscopic terms, the porosity percentage in the compact bone is 3-5%, whereas in the cancellous or trabecular the porosity percentage is up to 90% [1].

Mahmoud [1, 5, 6] investigated the wave propagation under the effects of initial stress, rotation and magnetic field in cylindrical poroelastic bones, a granular medium and a porous medium. Theoretical analyses of bone piezoelectricity may be relevant to the issue of bone remodeling. Recent thorough studies have explored electromechanical effects in wet and dry bone [7, 8]. They suggest that two different mechanisms are responsible for these effects: classical piezoelectricity mainly due to the molecular asymmetry of collagen in dry bone and streaming potentials found in moist or living bone and generated by the flow of a liquid across charged surfaces. The second mechanism was argued by dielectric measurements, and it was suggested that the electromechanical effect in wet (fluid saturated) bone is not due to a piezoelectric effect [9]. Abd-Alla and Mahmoud [10, 11] solved a magneto-thermoelastic problem in a rotating non-homogeneous orthotropic hollow cylinder under the hyperbolic heat conduction model and investigated analytical solution of wave propagation in non-homogeneous orthotropic rotating elastic media. Abd-Alla *et al.* [12] studied the propagation of S-wave in a non-homogeneous anisotropic incompressible and initially stressed medium under the influence of gravity field. Honarvarla *et al.* [13], Ding *et al.* [14] studied the elasticity of transversely isotropic materials. Chen *et al.* [15, 16] investigated the free vibration and general solution of non-homogeneous transversely isotropic magneto-electroelastic hollow cylinders. Abd-Alla *et al.* [17, 18] studied the problem of transient coupled thermoelasticity of an annular fin and the problem of radial vibrations in a non-homogeneous isotropic cylinder under the influence of initial stress and magnetic field. Mofakhami *et al.* [19] studied the finite cylinder vibrations with different end boundary conditions. Abd-Alla *et al.* [20, 21] studied the effect of rotation, magnetic field and initial stress on peristaltic motion of micropolar fluid and investigated the effect of rotation on a non-homogeneous infinite cylinder of orthotropic material.

In this paper, the equations of elastodynamics for transversely isotropic material under the effect of initial stress and magnetic field are solved in terms of displacement potentials. Also, this paper is concerned with the determination of phase velocity and the eigenvalues of natural frequency of plane vibrations of bones under the effect of initial stress and magnetic field for different boundary conditions in the cases of harmonic vibrations. The numerical results of the frequency equation are discussed in detail for transversely isotropic material and the effect of initial stress and magnetic field for different cases is indicated by figures.

2 Formulation of the problem

Consider a homogeneous and transversely isotropic long bone as a hollow cylinder of inner radius a and outer radius b taking the cylindrical polar coordinates (r,θ,z) such that the z-axis points vertically upward along the bone axis.

The equations of the elastodynamic medium in the presence of magnetic field are as follows:

$$\frac{\partial\tau_{rr}}{\partial r} + r^{-1}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + r^{-1}(\tau_{rr} - \tau_{\theta\theta}) + (\overrightarrow{J} \times \overrightarrow{B})_r = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{1a}$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + r^{-1}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + 2r^{-1}\tau_{\theta r} + (\overrightarrow{J} \times \overrightarrow{B})_\theta = \rho\frac{\partial^2 u_\theta}{\partial t^2}, \tag{1b}$$

$$\frac{\partial\tau_{rz}}{\partial r} + r^{-1}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + r^{-1}\tau_{\theta r} + (\overrightarrow{J} \times \overrightarrow{B})_z = \rho\frac{\partial^2 u_z}{\partial t^2}, \tag{1c}$$

where ρ is the density of bones, τ_{rr}, $\tau_{\theta\theta}$, τ_{zz} and τ_{rz} are the stresses, u_r, u_θ and u_z are the displacement components and $\overrightarrow{B} = \mu_e\overleftarrow{H}$, where μ_e is the magnetic permeability, $\overleftarrow{H} = (0,H_0,0)$ and H_0 is the intensity of the uniform magnetic field parallel to θ-axes, $\overleftarrow{F}$ is

Lorentz's body forces vector where $\overleftarrow{F}$ can take the following form:

$$\overleftarrow{F} = (f_r, f_\theta, f_z) = \big((\overrightarrow{J} \times \overrightarrow{B})_r, (\overrightarrow{J} \times \overrightarrow{B})_\theta, (\overrightarrow{J} \times \overrightarrow{B})_z\big). \tag{2}$$

The relations of stresses-displacement for homogeneous transversely isotropic bone in two dimensions are in the form:

$$\tau_{rr} = \left(c_{11} + P^*\right)\frac{\partial u_r}{\partial r} + \left(c_{12} + P^*\right)\frac{u_r}{r} + \left(c_{13} + P^*\right)\frac{\partial u_z}{\partial z}, \tag{3a}$$

$$\tau_{\theta\theta} = \left(c_{12} + P^*\right)\frac{\partial u_r}{\partial r} + \left(c_{11} + P^*\right)\frac{u_r}{r} + \left(c_{13} + P^*\right)\frac{\partial u_z}{\partial z}, \tag{3b}$$

$$\tau_{zz} = c_{13}\frac{\partial u_r}{\partial r} + c_{13}\frac{u_r}{r} + c_{33}\frac{\partial u_z}{\partial z}, \tag{3c}$$

$$\tau_{rz} = c_{44}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \tag{3d}$$

where c_{11}; c_{12}; c_{22}; c_{13}; c_{23}; c_{44} are the elastic constants of bone, $c_{11} = c_{22}$, $c_{13} = c_{23}$, and P^* is the initial stress compression. The electromagnetic field is governed by Maxwell's equations considering that the medium is a perfect electric conductor taking into account the absence of the displacement current SI.

$$\begin{gathered} \operatorname{curl} \overleftarrow{h} = \overrightarrow{J}, \qquad \operatorname{curl} \overleftarrow{E} = -\mu_e \frac{\partial \overleftarrow{h}}{\partial t}, \qquad \operatorname{div} \overleftarrow{h} = 0, \qquad \operatorname{div} \overleftarrow{E} = 0, \\ \overrightarrow{h} = \operatorname{curl}(\overrightarrow{u} \times \overrightarrow{H}_0), \qquad \overleftarrow{H} = \overleftarrow{H}_0 + \overleftarrow{h}, \qquad \overleftarrow{H}_0 = (0, H_0, 0). \end{gathered} \tag{4}$$

Maxwell's stresses $\bar{\tau}_{ij}$ take the following form:

$$\bar{\sigma}_{ij} = \mu_e\big[H_i h_j + H_j h_i - (\overrightarrow{H} \cdot \overrightarrow{h})\delta_{ij}\big], \quad i, j = 1, 2, 3. \tag{5}$$

Two-dimentional equations of elastodynamics are as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + H_0^2 \mu_e r^{-1}\left[\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + r\frac{\partial^2 u_z}{\partial r\,\partial z} + r\frac{\partial^2 u_r}{\partial r^2}\right] = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{6a}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} + H_0^2 \mu_e\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_r}{\partial r\,\partial z}\right) = \rho\frac{\partial^2 u_z}{\partial t^2}. \tag{6b}$$

Substituting equations (3a)-(3d) into equations (6a) and (6b), we obtain

$$\begin{aligned} &\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r + \frac{c_{44}}{c_{11}}\frac{\partial^2 u_r}{\partial z^2} + \frac{c_{13} + c_{44}}{c_{11}}\frac{\partial^2 u_z}{\partial r\,\partial z} + \frac{1}{2}P^*\frac{\partial(\frac{\partial u_r}{\partial z} - u_z)}{\partial z} \\ &\quad + H_0^2 \mu_e r^{-1}\left[\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + r\frac{\partial^2 u_z}{\partial r\,\partial z} + r\frac{\partial^2 u_r}{\partial r^2}\right] = \frac{\rho}{c_{11}}\left(\frac{\partial^2 u_r}{\partial t^2}\right), \end{aligned} \tag{7a}$$

$$\begin{aligned} &\frac{\partial^2 u_z}{\partial z^2} + \frac{c_{44}}{c_{33}}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + \frac{c_{13} + c_{44}}{c_{33}}\left(\frac{\partial^2 u_r}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) + \frac{P^*}{2r}\frac{\partial(r(\frac{\partial u_r}{\partial z} - u_z))}{\partial z} \\ &\quad + H_0^2 \mu_e\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_r}{\partial r\,\partial z}\right) = \frac{\rho}{c_{33}}\left(\frac{\partial^2 u_z}{\partial t^2}\right). \end{aligned} \tag{7b}$$

System (7a)-(7b) can be simplified as follows:

$$\frac{1}{2r^2}\left[-2c_{11}u_r + r^2H_0^2\mu_e\frac{\partial u_z}{\partial z} + 2\left(c_{11}+H_0^2\mu_e\right)\frac{\partial u_r}{\partial r} + r - 2\rho\frac{\partial^2 u_r}{\partial t^2} + (c_{44}+P)\frac{\partial^2 u_r}{\partial z^2}\right.$$
$$\left. + \left(2c_{13}+c_{44}-P+2H_0^2\mu_e\right)\frac{\partial^2 u_z}{\partial r\,\partial z} + 2\left(c_{11}+H_0^2\mu_e\right)\frac{\partial^2 u_r}{\partial r^2}\right] = 0, \tag{8a}$$

$$\frac{1}{2r}\left[(2c_{13}+c_{44}+P)\frac{\partial u_r}{\partial z} + (c_{44}-P)\frac{\partial u_z}{\partial r} + r\left(-2\rho\frac{\partial^2 u_z}{\partial t^2} + 2\left(c_{33}+H_0^2\mu_e\right)\frac{\partial^2 u_z}{\partial z^2}\right.\right.$$
$$\left.\left. + \left(2c_{13}+c_{44}+P+2H_0^2\mu_e\right)\frac{\partial^2 u_r}{\partial r\,\partial z} + (c_{44}-P)\frac{\partial^2 u_z}{\partial r^2}\right)\right] = 0. \tag{8b}$$

3 Solution of the problem

By Helmohltz's theorem [22] the displacement vector $\overleftarrow{u}$ can be written as

$$\overleftarrow{u} = \underline{\nabla}\phi_1 + \underline{\nabla}\wedge\overleftarrow{\Psi}, \tag{9}$$

where the two functions ϕ_1 and $\overleftarrow{\Psi} = (0,\psi_1,0)$ are known in the theory of elasticity, by Lame's potentials irrotational and rotational parts of the displacement vector $\overleftarrow{u}$, respectively. The cylinder being bounded by the curved surface, the stress distribution includes the effect of both ϕ_1 and $\overleftarrow{\psi}$. It is possible to take only one component of the vector $\overleftarrow{\psi}$ to be nonzero as follows: $\overleftarrow{\psi} = (0,\psi_1,0)$. From (9) we obtain

$$u_r = \frac{\partial\phi_1}{\partial r} - \frac{\partial\psi_1}{\partial z}, \tag{10a}$$

$$u_z = \frac{\psi_1}{r} + \frac{\partial\phi}{\partial z} + \frac{\partial\psi_1}{\partial r}. \tag{10b}$$

Substituting equations (10a) and (10b) into equations (8a) and (8b), we get two independent equations for ϕ_1 and ψ_1 as follows:

$$\frac{1}{2r^2}\left[2c_{11}\left(\frac{\partial\psi_1}{\partial z} - \frac{\partial\phi_1}{\partial r}\right) + r\left(2H_0^2\mu_e\left(\frac{\partial\psi_1}{\partial z}r^{-1} + \frac{\partial^2\phi_1}{\partial z^2} + \frac{\partial^2\psi_1}{\partial r\,\partial z}\right)\right.\right.$$
$$+ 2\left(c_{11}+H_0^2\mu_e\right)\left(-\frac{\partial^2\psi_1}{\partial r\,\partial z} + \frac{\partial^2\phi_1}{\partial r^2}\right) + r\left(2\rho\left(\frac{\partial^3\psi_1}{\partial z\,\partial t^2} - \frac{\partial^3\phi_1}{\partial r\,\partial t^2}\right)\right.$$
$$+ \left(c_{44}+P^*\right)\left(-\frac{\partial^3\psi_1}{\partial z^3} + \frac{\partial^3\phi_1}{\partial r\,\partial z^2}\right) + \left(2c_{13}+c_{44}-P^*+2H_0^2\mu_e\right)$$
$$\times\left(-r^{-2}\frac{\partial\psi_1}{\partial z} + r^{-1}\frac{\partial^2\psi_1}{\partial r\,\partial z} + \frac{\partial^3\phi_1}{\partial r\,\partial z^2} + \frac{\partial^3\psi_1}{\partial r^2\,\partial z}\right)$$
$$\left.\left.\left. + 2\left(c_{11}+H_0^2\mu_e\right)\left(-\frac{\partial^3\psi_1}{\partial r^2\partial z} + \frac{\partial^3\phi_1}{\partial r^3}\right)\right)\right)\right], \tag{11a}$$

$$\frac{1}{2r}\left(2c_{13}+c_{44}+P^*\right)\left(-\frac{\partial^2\psi_1}{\partial z^2} + \frac{\partial^2\phi_1}{\partial r\,\partial z}\right)$$
$$+ \left(c_{44}-P^*\right)\left(-\psi_1 r^{-2} + \frac{\partial\psi_1}{\partial r}r^{-2} + \frac{\partial^2\phi_1}{\partial r\,\partial z} + \frac{\partial^2\psi_1}{\partial r^2}\right)$$
$$+ r\left(-2\rho\left(\frac{\partial^2\psi_1}{\partial t^2}r^{-1} + \frac{\partial^3\phi_1}{\partial z\,\partial t^2} + \frac{\partial^3\psi_1}{\partial r\,\partial t^2}\right) + 2\left(c_{33}+H_0^2\mu_e\right)\right.$$

$$\times \left(\frac{\partial^2 \psi_1}{\partial z^2} r^{-1} + \frac{\partial^3 \phi_1}{\partial z^3} + \frac{\partial^3 \psi_1}{\partial r\, \partial z^2} \right) + \left(2c_{13} + c_{44} + P^* + 2H_0^2 \mu_e\right) \left(-\frac{\partial^3 \psi_1}{\partial r\, \partial z^2} + \frac{\partial^3 \phi_1}{\partial r^2\, \partial z} \right)$$
$$+ \left(c_{44} - P^*\right) \left(\left(2\psi_1 - 2r \frac{\partial \psi_1}{\partial r} + r2 \frac{\partial^2 \psi_1}{\partial r^2} \right) r^{-3} + \frac{\partial^3 \phi_1}{\partial r^2\, \partial z} + \frac{\partial^3 \psi_1}{\partial r^3} \right) \Bigg) = 0. \qquad \text{(11b)}$$

To study the propagation of harmonic waves in the z-direction, we assume a solution in the form

$$\Phi_1(r,z,t) = \Phi_2(r) e^{i(\gamma z - \omega_5 t)}, \qquad \text{(12a)}$$

$$\Psi_1(r,z,t) = \Psi_2(r) e^{i(\gamma z - \omega_5 t)}, \qquad \text{(12b)}$$

where γ is the wave number, ω is the angular frequency. Substituting equations (12a) and (12b) into equations (11a) and (11b) and omitting the factor exponential throughout, we have

$$-\frac{1}{2r^2} i e^{i(z\gamma - t\omega_5)} \Bigg[-2iH_0^2 r\gamma^2 \mu_e \Phi_2(r) - \gamma \left(2c_{11} - 2c_{13} + P^* + c_{44}\left(-1 + r^2\gamma^2\right)\right.$$
$$\left. + r^2\left(P^*\gamma^2 - 2\rho\omega_5^2\right)\right) \Psi_2(r) - 2ic_{11} - 2ic_{13} r^2 \gamma^2 \frac{d\Phi_2}{dr} - 2ic_{44} r^2 \gamma \frac{d\Phi_2}{dr} 2$$
$$- 2iH_0^2 r^2 \gamma^2 \mu_e \frac{d\Phi_2}{dr} + 2ir^2 \rho\omega_5^2 \frac{d\Phi_2}{dr} + 2c_{11} r\gamma \frac{d\Psi_2}{dr} - 2c_{13} r\gamma \frac{d\Psi_2}{dr} - c_{14} r\gamma \frac{d\Psi_2}{dr}$$
$$+ P^* r\gamma \frac{d\Psi_2}{dr} - 2H_0^2 r\gamma \mu_e \frac{d\Psi_2}{dr} + 2ic_{11} r \frac{d^2\Phi_2}{dr^2} + 2iH_0^2 r\mu_e \frac{d^2\Phi_2}{dr^2} + 2c_{11} r^2 \gamma \frac{d^2\Psi_2}{dr^2}$$
$$- 2c_{13} r^2 \gamma \frac{d^2\Psi_2}{dr^2} - c_{44} r^2 \gamma \frac{d^2\Psi_2}{dr^2} + P^* r^2 \gamma \frac{d^2\Psi_2}{dr^2}$$
$$+ 2ir^2\left(c_{11} + H_0^2\mu_e\right) \frac{d^3\Phi_2}{dr^3} \Bigg] = 0, \qquad \text{(13a)}$$

$$\frac{1}{2r^3} e^{i(z\gamma - t\omega_5)} \Bigg[-2ir^3\gamma\left(\gamma^2\left(c_{33} + H_0^2\mu_e\right) - \rho\omega_5^2\right)\Phi_2(r) + \left(c_{44} + c_{44} r^2\gamma^2\right.$$
$$\left. + P^*\left(-1 + r^2\gamma^2\right) + 2r^2\left(\gamma^2\left(c_{13} - c_{33} - H_0^2\mu_e\right) + \rho\omega_5^2\right)\right)\Psi_2(r)$$
$$+ r\Bigg(2i(c_{13} + c_{44}) r\gamma \frac{d\Phi_2}{dr} + \left(P^* + P^* r^2\gamma^2 + c_{44}\left(-1 + r^2\gamma^2\right)\right.$$
$$\left. + 2r^2\left((c_{13} - c_{33})\gamma^2 + \rho\omega_5^2\right)\right) \frac{d\Psi_2}{dr} + r\Bigg(2ir\gamma\left(c_{13} + c_{44} + H_0^2\mu_e\right) \frac{d^2\Phi_2}{dr^2}$$
$$+ \left(c_{44} - P^*\right)\left(2\frac{d\Psi_2}{dr} + r\frac{d^3\Psi_2}{dr^3} \right)\Bigg)\Bigg)\Bigg] = 0. \qquad \text{(13b)}$$

Similar results were obtained by Elnagar and Abd-Alla [23], the former deriving the constitutive equation for Rayleigh waves in an elastic medium under initial stress, and the latter deriving the constitutive equation for thermoelastic problems in an infinite cylinder under initial stress. From equations (13a) and (13b), we get equations (14a) and (14b) as follows:

$$2\Bigg(-H_0^2 r\gamma^2\mu_e\Phi_2(r) - \left(c_{11} + r^2\left(\gamma^2\left(c_{13} + c_{44} + H_0^2\mu_e\right) - \rho\omega_5^2\right)\right) \frac{d\Phi_2}{dr}$$
$$+ r\left(c_{11} + H_0^2\mu_e\right)\left(\frac{d^2\Phi_2}{dr^2} + r\frac{d^3\Phi_2}{dr^3} \right)\Bigg) = 0, \qquad \text{(14a)}$$

$$\begin{aligned}
&\left(c_{44} + c_{44}r^2\gamma^2 + P^*\left(-1 + r^2\gamma^2\right) + 2r^2\left(\gamma^2\left(c_{13} - c_{33} - H_0^2\mu_e\right) + \rho\omega_5^2\right)\right)\Psi_2(r) \\
&\quad + r\left(\left(P^* + P^*r^2\gamma^2 + c_{44}\left(-1 + r^2\gamma^2\right) + 2r^2\left((c_{13} - c_{33})\gamma^2 + \rho\omega_5^2\right)\right)\frac{d\Psi_2}{dr} \right.\\
&\quad \left. + \left(c_{44} - P^*\right)r\left(2\frac{d^2\Psi_2}{dr^2} + r\frac{d^3\Psi_2}{dr^3}\right)\right) = 0.
\end{aligned} \tag{14b}$$

Equation (14a) represents the shear wave, and equation (14b) represents the longitudinal wave. The solution of equations (14a) and (14b) can be written in the following form:

$$\Psi_2 = A_1 r^{d_1} J_{n_1}(d_1 r) + B_1 r^{d_1} Y_{n_1}(d_2 r), \tag{15a}$$

$$\Phi_2 = A_2 J_{n_2}(e_1 r) + B_2 Y_{n_2}(e_2 r), \tag{15b}$$

where

$$\begin{aligned}
n_1 &= \frac{\sqrt{d_0 - 4c_{11}c_{44} + 4c_{13}c_{44} + c_{44}^2 + 4c_{11}P^* - 4c_{13}P^* - 2c_{44}P^* + P^{*2} + H_0^4\mu_e^2}}{2c_{11} - 2c_{13} - c_{44} + P^*}, \\
d_1 &= \frac{H_0^2\mu_e}{2c_{11} - 2c_{13} - c_{44} + P}, \qquad d_2 = -\frac{ir\sqrt{c_{44}\gamma^2 + P^*\gamma^2 - 2\rho\omega_5^2}}{\sqrt{2c_{11} - 2c_{13} - c_{44} + P^*}}, \\
e_1 &= \frac{H_0^2\mu_e}{2(c_{13} + c_{44} + H_0^2\mu_e)}, \qquad d_0 = 4c_{11}^2 - 8c_{11}c_{13} + 4c_{13}^2, \qquad n_2 = \frac{H_0^2\mu_e}{2(c_{13} + c_{44} + H_0^2\mu_e)}, \\
e_2 &= \frac{\sqrt{c_{33}\gamma 2 + H_0^2\gamma 2\mu_e - \rho\omega_5^2}}{\sqrt{c_{13} + c_{44} + H_0^2\mu_e}}.
\end{aligned}$$

A_1, A_2, B_1 and B_2 are arbitrary constants, $J_\circ$ is the Bessel function of the first kind and of order zero, $Y_\circ$ is the Bessel function of the second kind and of order zero. J_n is the Bessel function of the first kind and of order n, Y_n is the Bessel function of the second kind and of order n. From equations (12a), (12b) and (15a), (15b) we get

$$\Psi_1(r,z,t) = e^{i(z\gamma - t\omega_5)}\left(A_1 r^{d_1} J_{n_1}(d_1 r) + B_1 r^{d_2} Y_{n_1}(d_2 r)\right), \tag{16a}$$

$$\Phi_1(r,z,t) = e^{i(z\gamma - t\omega_5)}\left(A_2 J_{n_2}(e_1 r) + B_2 Y_{n_2}(e_2 r)\right). \tag{16b}$$

Substituting equations (16a) and (16b) into equations (10a) and (10b), we obtain the final solution of displacement components in the following form:

$$\begin{aligned}
u_r = \frac{1}{r}e^{i(z\gamma - t\omega_5)}&\left[A_2 e_1 J_{-1+n_2}(e_1 r) - J_{1+n_2}(e_1 r) - 2ir^{d_1}\gamma A_1 J_{n_1}(d_2 r) + B_1 Y_{n_1}(d_2 r)\right. \\
&\left. + B_2 e_2 Y_{-1+n_2}(e_2 r) - Y_{1+n_2}(e_2 r)\right],
\end{aligned} \tag{17a}$$

$$\begin{aligned}
u_z = \frac{1}{r}e^{i(z\gamma - t\omega_5)}&\left[A_1 d_2 r^{1+d_1} J_{-1+n_1}(d_2 r) + A_1(1 + d_1 - n_1)r^{d_1} J_{n_1}(d_2 r) + iA_2 r\gamma J_{n_2}(e_1 r)\right. \\
&\left. + B_1 d_2 r^{1+d_1} Y_{-1+n_1}(d_2 r) + B_1(1 + d_1 - n_1)r^{d_1} Y_{n_1}(d_2 r) + iB_2 r\gamma Y_{n_2}(e_2 r)\right].
\end{aligned} \tag{17b}$$

Substituting equations (17a) and (17b) into equations (3a)-(3d), we obtain the final solution of the stress components of solid in the following form:

$$\begin{aligned}
\tau_{rr} = {}& \frac{1}{4r}e^{i(z\gamma - t\omega_5)}\big[A_2c_{11}e_1^2rJ_{-2+n_2}(e_1r) + 2A_2c_{12}e_1J_{-1+n_2}(e_1r) - 2A_2c_{11}e_1^2rJ_{n_2}(e_1r) \\
& - 4A_2c_{13}r\gamma^2J_{n_2}(e_1r) - 2A_2c_{12}e_1J_{1+n_2}(e_1r) + A_2c_{11}e_1^2rJ_{2+n_2}(e_1r) \\
& - 4ird_1\gamma\big(A_1(c_{11}-c_{13})d_2rJ_{-1+n_1}(d_2r) + A_1\big(c_{12} + c_{11}(d_1-n_1) \\
& + c_{13}(-1-d_1+n_1)\big)J_{n_1}(d_2r) + B_1(c_{11}-c_{13})d_2rY_{-1+n_1}(d_2r) \\
& + B_1\big(c_{12} + c_{11}(d_1-n_1) + c_{13}(-1-d_1+n_1)\big)Y_{n_1}(d_2r)\big) + B_2c_{11}e_2^2rY_{-2+n_2}(e_2r) \\
& + 2B_2c_{12}e_2Y_{-1+n_2}(e_2r) - 2B_2c_{11}e_2^2rY_{n_2}(e_2r) - 4B_2c_{13}r\gamma^2Y_{n_2}(e_2r) \\
& - 2B_2c_{12}e_2Y_{1+n_2}(e_2r) + B_2c_{11}e_2^2rY_{2+n_2}(e_2r)\big],
\end{aligned} \tag{18a}$$

$$\begin{aligned}
\tau_{\theta\theta} = {}& \frac{1}{4r}e^{i(z\gamma - t\omega_5)}\big[A_2c_{12}e_1^2rJ_{-2+n_2}(e_1r) + 2A_2c_{11}e_1J_{-1+n_2}(e_1r) - 2A_2c_{12}e_1^2rJ_{n_2}(e_1r) \\
& - 4A_2c_{13}r\gamma^2J_{n_2}(e_1r) - 2A_2c_{11}e_1J_{1+n_2}(e_1r) + A_2c_{12}e_1^2rJ_{2+n_2}(e_1r) \\
& - 4ird_1\gamma\big(A_1(c_{12}-c_{13})d_2rJ_{-1+n_1}(d_2r) + A_1\big(c_{11} + c_{12}(d_1-n_1) \\
& + c_{13}(-1-d_1+n_1)\big)J_{n_1}(d_2r) + B_1(c_{12}-c_{13})d_2rY_{-1+n_1}(d_2r) \\
& + B_1\big(c_{11} + c_{12}(d_1-n_1) + c_{13}(-1-d_1+n_1)\big)Y_{n_1}(d_2r)\big) + B_2c_{12}e_2^2rY_{-2+n_2}(e_2r) \\
& + 2B_2c_{11}e_2Y_{-1+n_2}(e_2r) - 2B_2c_{12}e_2^2rY_{n_2}(e_2r) - 4B_2c_{13}r\gamma^2Y_{n_2}(e_2r) \\
& - 2B_2c_{11}e_2J_{1+n_2}(e_2r) + B_2c_{12}e_2^2rY_{2+n_2}(e_2r)\big],
\end{aligned} \tag{18b}$$

$$\begin{aligned}
\tau_{zz} = {}& \frac{1}{r}e^{i(z\gamma - t\omega_5)}\big[-i(c_{13}-c_{33})rd_1\gamma\big(A_1d_2rJ_{-1+n_1}(d_2r) + A_1(1+d_1-n_1)J_{n_1}(d_2r) \\
& + B_1d_2rY_{-1+n_1}(d_2r) + B_1(1+d_1-n_1)Y_{n_1}(d_2r)\big) \\
& + r^{-1}\big(A_2\big(c_{13}(n_2-e_1r)(n_2+e_1r) - c_{33}r^2\gamma^2\big)J_{n_2}(e_1r) \\
& + B_2\big(c_{13}(n_2-e_2r)(n_2+e_2r) - c_{33}r^2\gamma^2\big)Y_{n_2}(e_2r)\big)\big],
\end{aligned} \tag{18c}$$

$$\begin{aligned}
\tau_{rz} = {}& \frac{1}{2r^2}c_{44}e^{i(z\gamma - t\omega_5)}\big[-A_1r^{d_1}\big(-1 + (d_1-n_1)^2 + r^2\big(-d_2^2+\gamma^2\big)\big)J_{-2+n_1}(d_2r) \\
& + 2A_1r^{-1+d_1}\big(d_1^2(-1+n_1) + d_1\big(-2(-1+n_1)n_1 + d_2^2r^2\big) \\
& + (-1+n_1)\big(-1+n_1^2+r^2\big(-d_2^2+\gamma^2\big)\big)\big)J_{-1+n_1}(d_2r)d_2 + 2iA_2e_1r^2\gamma J_{-1+n_2}(e_1r) \\
& - 2iA_2n_2r\gamma J_{n_2}(e_1r) - B_1rd_1\big(-1 + (d_1-n_1)^2 + r^2\big(-d_2^2+\gamma^2\big)\big)Y_{-2+n_1}(d_2r) \\
& + 2B_1r^{-1+d_1}\big(d_1^2(-1+n_1) + d_1\big(-2(-1+n_1)n_1 + d_2^2r^2\big) \\
& + (-1+n_1)\big(-1+n_1^2+r^2\big(-d_2^2+\gamma^2\big)\big)\big)Y_{-1+n_1}(d_2r)d_2 + 2iB_2e_2r^2\gamma Y_{-1+n_2}(e_2r) \\
& - 2iB_2n_2r\gamma Y_{n_2}(e_2r)\big],
\end{aligned} \tag{18d}$$

where J_1 and Y_1 are the Bessel functions of the first order. In the following section, solutions of hollow circular cylinders with three different boundary conditions are performed.

4 Boundary conditions and frequency equation

In this case, we are going to obtain the frequency equation for the boundary conditions. Plane vibrations cylindrical bone free surface traction. In this case, we have

$$\tau_{rr}(r)+\sigma_{rr}(r)=0,\qquad \tau_{rz}(r)=0,\quad \text{at } r=a, \tag{19a}$$

$$\tau_{rr}(r)+\sigma_{rr}(r)=0,\qquad \tau_{rz}(r)=0,\quad \text{at } r=b, \tag{19b}$$

which correspond to free inner and outer surfaces, respectively. From equations (18a)-(18d), (5) and (19a)-(19b) we obtain four homogeneous linear equations in A_1, B_1, A_2 and B_2

$$\begin{aligned}
&A_2c_{11}e_1^2rJ_{-2+n_2}(e_1a)+2A_2c_{12}e_1J_{-1+n_2}(e_1a)-2A_2c_{11}e_1^2aJ_{n_2}(e_1a)\\
&\quad-4A_2c_{13}a\gamma^2J_{n_2}(e_1a)-2A_2c_{12}e_1J_{1+n_2}(e_1a)+A_2c_{11}e_1^2aJ_{2+n_2}(e_1a)\\
&\quad-4iad_1\gamma\bigl(A_1(c_{11}-c_{13})d_2aJ_{-1+n_1}(d_2a)+A_1\bigl(c_{12}+c_{11}(d_1-n_1)\\
&\quad+c_{13}(-1-d_1+n_1)\bigr)J_{n_1}(d_2a)+B_1(c_{11}-c_{13})d_2aY_{-1+n_1}(d_2a)\\
&\quad+B_1\bigl(c_{12}+c_{11}(d_1-n_1)+c_{13}(-1-d_1+n_1)\bigr)Y_{n_1}(d_2a)\bigr)+B_2c_{11}e_2^2aY_{-2+n_2}(e_2a)\\
&\quad+2B_2c_{12}e_2Y_{-1+n_2}(e_2a)-2B_2c_{11}e_2^2aY_{n_2}(e_2a)-4B_2c_{13}a\gamma^2Y_{n_2}(e_2a)\\
&\quad-2B_2c_{12}e_2Y_{1+n_2}(e_2a)+B_2c_{11}e_2^2aY_{2+n_2}(e_2a)=0,
\end{aligned}\tag{20a}$$

$$\begin{aligned}
&-A_1a^{d_1}\bigl(-1+(d_1-n_1)^2+a^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)J_{-2+n_1}(d_2a)+2A_1a^{-1+d_1}\bigl(d_1^2(-1+n_1)\\
&\quad+d_1\bigl(-2(-1+n_1)n_1+d_2^2a^2\bigr)+(-1+n_1)\bigl(-1+n_1^2+a^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)\bigr)J_{-1+n_1}(d_2a)d_2\\
&\quad+2iA_2e_1a^2\gamma J_{-1+n_2}(e_1a)-2iA_2n_2a\gamma J_{n_2}(e_1a)-B_1ad_1\bigl(-1+(d_1-n_1)^2\\
&\quad+a^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)Y_{-2+n_1}(d_2a)+2B_1a^{-1+d_1}\bigl(d_1^2(-1+n_1)+d_1\bigl(-2(-1+n_1)n_1+d_2^2a^2\bigr)\\
&\quad+(-1+n_1)\bigl(-1+n_1^2+a^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)\bigr)Y_{-1+n_1}(d_2a)d_2\\
&\quad+2iB_2e_2a^2\gamma Y_{-1+n_2}(e_2a)-2iB_2n_2a\gamma Y_{n_2}(e_2a)=0,
\end{aligned}\tag{20b}$$

$$\begin{aligned}
&A_2c_{11}e_1^2bJ_{-2+n_2}(e_1b)+2A_2c_{12}e_1J_{-1+n_2}(e_1b)-2A_2c_{11}e_1^2bJ_{n_2}(e_1b)-4A_2c_{13}b\gamma^2J_{n_2}(e_1b)\\
&\quad-2A_2c_{12}e_1J_{1+n_2}(e_1b)+A_2c_{11}e_1^2bJ_{2+n_2}(e_1b)-4ibd_1\gamma\bigl(A_1(c_{11}-c_{13})d_2bJ_{-1+n_1}(d_2b)\\
&\quad+A_1\bigl(c_{12}+c_{11}(d_1-n_1)+c_{13}(-1-d_1+n_1)\bigr)J_{n_1}(d_2b)+B_1(c_{11}-c_{13})d_2bY_{-1+n_1}(d_2b)\\
&\quad+B_1\bigl(c_{12}+c_{11}(d_1-n_1)+c_{13}(-1-d_1+n_1)\bigr)Y_{n_1}(d_2b)\bigr)+B_2c_{11}e_2^2bY_{-2+n_2}(e_2b)\\
&\quad+2B_2c_{12}e_2Y_{-1+n_2}(e_2b)-2B_2c_{11}e_2^2bY_{n_2}(e_2b)-4B_2c_{13}b\gamma^2Y_{n_2}(e_2b)\\
&\quad-2B_2c_{12}e_2Y_{1+n_2}(e_2b)+B_2c_{11}e_2^2bY_{2+n_2}(e_2b)=0,
\end{aligned}\tag{20c}$$

$$\begin{aligned}
&-A_1b^{d_1}\bigl(-1+(d_1-n_1)^2+b^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)J_{-2+n_1}(d_2b)+2A_1b^{-1+d_1}\bigl(d_1^2(-1+n_1)\\
&\quad+d_1\bigl(-2(-1+n_1)n_1+d_2^2b^2\bigr)+(-1+n_1)\bigl(-1+n_1^2+b^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)\bigr)J_{-1+n_1}(d_2b)d_2\\
&\quad+2iA_2e_1b^2\gamma J_{-1+n_2}(e_1b)-2iA_2n_2b\gamma J_{n_2}(e_1b)-B_1bd_1\bigl(-1+(d_1-n_1)^2\\
&\quad+b^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)Y_{-2+n_1}(d_2b)+2B_1b^{-1+d_1}\bigl(d_1^2(-1+n_1)+d_1\bigl(-2(-1+n_1)n_1+d_2^2b^2\bigr)\\
&\quad+(-1+n_1)\bigl(-1+n_1^2+b^2\bigl(-d_2^2+\gamma^2\bigr)\bigr)\bigr)Y_{-1+n_1}(d_2b)d_2\\
&\quad+2iB_2e_2b^2\gamma Y_{-1+n_2}(e_2b)-2iB_2n_2b\gamma Y_{n_2}(e_2b)=0.
\end{aligned}\tag{20d}$$

These are a set of four homogeneous algebraic equations involving four unknown integration constants A_1, B_1, A_2 and B_2. For a nontrivial solution of these equations, the determinant of the coefficient matrix must vanish. The zero determinant of the coefficient matrix will give the frequency equation for the surface waves. Thus, elimination of these unknowns would give us the frequency equation as follows:

$$\Delta = \begin{vmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{vmatrix} = 0. \tag{21}$$

The roots of equation (21) give the values of natural frequency for the free surfaces of the cylinder.

$$\begin{aligned}
B_{11} &= -4iad_1\gamma\big((c_{11}-c_{13})d_2aJ_{-1+n_1}(d_2a) + \big(c_{12} + c_{11}(d_1-n_1) \\
&\qquad + c_{13}(-1-d_1+n_1)\big)J_{n_1}(d_2a)\big), \\
B_{12} &= c_{11}e_1^2rJ_{-2+n_2}(e_1a) + 2c_{12}e_1J_{-1+n_2}(e_1a) - 2c_{11}e_1^2aJ_{n_2}(e_1a) - 4c_{13}a\gamma^2J_{n_2}(e_1a) \\
&\qquad - 2c_{12}e_1J_{1+n_2}(e_1a) + c_{11}e_1^2aJ_{2+n_2}(e_1a), \\
B_{13} &= -4iad_1\gamma\big(+B_1(c_{11}-c_{13})d_2aY_{-1+n_1}(d_2a) + B_1\big(c_{12} + c_{11}(d_1-n_1) \\
&\qquad + c_{13}(-1-d_1+n_1)\big)Y_{n_1}(d_2a)\big), \\
B_{14} &= c_{11}e_2^2aY_{-2+n_2}(e_2a) + 2c_{12}e_2Y_{-1+n_2}(e_2a) - 2c_{11}e_2^2aY_{n_2}(e_2a) - 4c_{13}a\gamma^2Y_{n_2}(e_2a) \\
&\qquad - 2c_{12}e_2Y_{1+n_2}(e_2a) + c_{11}e_2^2aY_{2+n_2}(e_2a), \\
B_{21} &= -a^{d_1}\big(-1 + (d_1-n_1)^2 + a^2\big(-d_2^2+\gamma^2\big)\big)J_{-2+n_1}(d_2a) + 2a^{-1+d_1}\big(d_1^2(-1+n_1) \\
&\qquad + d_1\big(-2(-1+n_1)n_1 + d_2^2a^2\big) + (-1+n_1)\big(-1+n_1^2+a^2\big(-d_2^2+\gamma^2\big)\big)\big)J_{-1+n_1}(d_2a)d_2, \\
B_{22} &= 2ie_1a^2\gamma J_{-1+n_2}(e_1a) - 2in_2a\gamma J_{n_2}(e_1a), \\
B_{23} &= -ad_1\big(-1 + (d_1-n_1)^2 + a^2\big(-d_2^2+\gamma^2\big)\big)Y_{-2+n_1}(d_2a) + 2a^{-1+d_1}\big(d_1^2(-1+n_1) \\
&\qquad + d_1\big(-2(-1+n_1)n_1 + d_2^2a^2\big) + (-1+n_1)\big(-1+n_1^2+a^2\big(-d_2^2+\gamma^2\big)\big)\big)Y_{-1+n_1}(d_2a)d_2, \\
B_{24} &= 2ie_2a^2\gamma Y_{-1+n_2}(e_2a) - 2in_2a\gamma Y_{n_2}(e_2a), \\
B_{31} &= -4ibd_1\gamma\big((c_{11}-c_{13})d_2bJ_{-1+n_1}(d_2b) + \big(c_{12} + c_{11}(d_1-n_1) \\
&\qquad + c_{13}(-1-d_1+n_1)\big)J_{n_1}(d_2b)\big), \\
B_{32} &= c_{11}e_1^2rJ_{-2+n_2}(e_1b) + 2c_{12}e_1J_{-1+n_2}(e_1b) - 2c_{11}e_1^2bJ_{n_2}(e_1b) - 4c_{13}b\gamma^2J_{n_2}(e_1b) \\
&\qquad - 2c_{12}e_1J_{1+n_2}(e_1b) + c_{11}e_1^2bJ_{2+n_2}(e_1b), \\
B_{33} &= -4ibd_1\gamma\big(+B_1(c_{11}-c_{13})d_2bY_{-1+n_1}(d_2b) + B_1\big(c_{12} + c_{11}(d_1-n_1) \\
&\qquad + c_{13}(-1-d_1+n_1)\big)Y_{n_1}(d_2b)\big), \\
B_{34} &= c_{11}e_2^2bY_{-2+n_2}(e_2b) + 2c_{12}e_2Y_{-1+n_2}(e_2b) - 2c_{11}e_2^2bY_{n_2}(e_2b) - 4c_{13}b\gamma^2Y_{n_2}(e_2b) \\
&\qquad - 2c_{12}e_2Y_{1+n_2}(e_2b) + c_{11}e_2^2bY_{2+n_2}(e_2b),
\end{aligned}$$

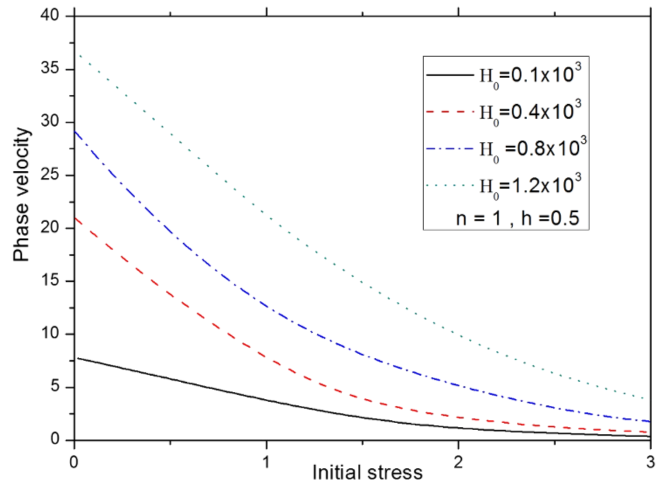

Figure 1 Variation of phase velocity with respect to initial stress P^* for different values of magnetic field H_0 for inner and outer free surfaces.

$$\begin{aligned}
B_{41} &= -b^{d_1}\big(-1 + (d_1 - n_1)^2 + b^2\big(-d_2^2 + \gamma^2\big)\big)J_{-2+n_1}(d_2 b) + 2b^{-1+d_1}\big(d_1^2(-1 + n_1) \\
&\quad + d_1\big(-2(-1 + n_1)n_1 + d_2^2 b^2\big) + (-1 + n_1)\big(-1 + n_1^2 + b^2\big(-d_2^2 + \gamma^2\big)\big)\big)J_{-1+n_1}(d_2 b)d_2, \\
B_{42} &= 2ie_1 b^2 \gamma J_{-1+n_2}(e_1 b) - 2in_2 b\gamma J_{n_2}(e_1 b), \\
B_{43} &= -bd_1\big(-1 + (d_1 - n_1)^2 + b^2\big(-d_2^2 + \gamma^2\big)\big)Y_{-2+n_1}(d_2 b) + 2b^{-1+d_1}\big(d_1^2(-1 + n_1) \\
&\quad + d_1\big(-2(-1 + n_1)n_1 + d_2^2 b^2\big) + (-1 + n_1)\big(-1 + n_1^2 + b^2\big(-d_2^2 + \gamma^2\big)\big)\big)Y_{-1+n_1}(d_2 b)d_2, \\
B_{44} &= 2ie_2 b^2 \gamma Y_{-1+n_2}(e_2 b) - 2in_2 b\gamma Y_{n_2}(e_2 b).
\end{aligned}$$

5 Numerical results and discussion

For the numerical calculation of dimensionless frequency and phase velocity under the effect of initial stress P^* and magnetic field H_0, one shall investigate the frequency equations given by (21) numerically for a particular model. Since these equations are an implicit function relation of dimensionless frequency, one can proceed with finding the variation of frequency with ratio h. Once the frequency has been computed, the corresponding effect of initial stress P^* and magnetic field H_0 on the frequency equations for dimensionless frequency (the eigenvalues) can be studied by taking values of ratio h. As an illustrative purpose of the foregoing solutions, the cylinder has the following geometric and material constants which are used in the computations given by [1, 2, 14]: $c_{11} = 2.33 \times 10^8$, $c_{12} = 0.85 \times 10^8$, $c_{13} = 1.3 \times 10^8$, $c_{44} = 0.62 \times 10^8$, $c_{33} = 4.981 \times 10^8$, $\rho = 3.986 \times 10^3$.

For various values of dimensionless frequency, the phase velocity is obtained from frequency equation flexural modes ($n = 1$). Here, we explain graphically our results of the previous applications. The dimensionless frequency and phase velocity were calculated with the aid of an electronic computer by using the half-interval method. Figure 1 shows the variation of phase velocity with respect to initial stress P^* for different values of magnetic field H_0 for inner and outer free surfaces. The phase velocity decreases with the increase in initial stress, but the phase velocity increases when the values of magnetic field H_0 increase. Figure 2 shows variation of the dimensionless frequency with respect to initial stress P^* for different values of magnetic field H_0 for inner and outer free surfaces. The

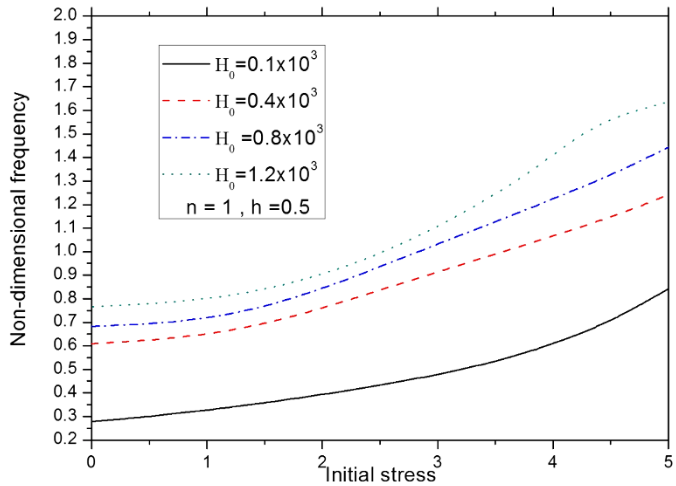

Figure 2 Variation of the dimensionless frequency with respect to initial stress P^* for different values of magnetic field H_0 for inner and outer free surfaces.

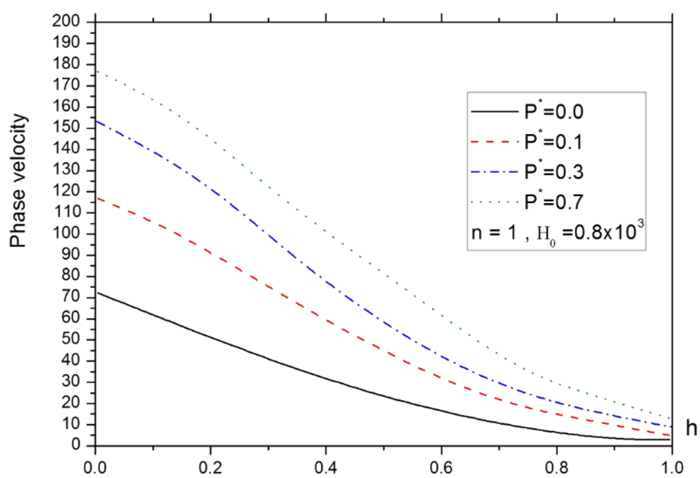

Figure 3 Variation of phase velocity with respect to the ratio (h) of inner and outer free surfaces for different values of initial stress P^* when magnetic field $H_0 = 0.8 \times 10^3$.

dimensionless frequency increases with the increase in initial stress, and the dimensionless frequency increases when the values of magnetic field H_0 increase. Figure 3 shows variation of phase velocity with respect to the ratio (h) of inner and outer free surfaces for different values of initial stress P^* when magnetic field $H_0 = 0.8 \times 10^3$. The phase velocity decreases with the increase in the ratio (h) of inner and outer free surfaces, and the phase velocity increases when the values of initial stress increase. Figure 4 shows variation of non-dimensional frequency with respect to the ratio (h) of inner and outer free surfaces for different values of initial stress P^* when magnetic field $H_0 = 0.8 \times 10^3$. The dimensionless frequency increases with the increase in the ratio (h) of inner and outer free surfaces, and the dimensionless frequency increases when the values of initial stress P^* decrease. It should be noticed that the effects of initial stress P^* and magnetic field H_0 on the dimensionless frequency tend to be the increasing dimensionless frequency. These results are

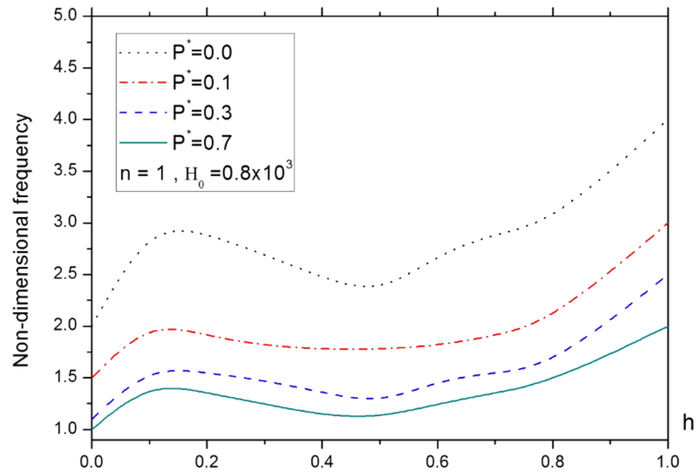

Figure 4 Variation of non-dimensional frequency with respect to the ratio (h) of inner and outer free surfaces for different values of initial stress P^* when magnetic field $H_0 = 0.8 \times 10^3$.

specific for the case considered, but other cases may have different trends because of the dependences of the results on the mechanical and constants of the material [24–26]. The results indicate that the effect of initial stress P^* and magnetic field H_0 is very pronouced.

6 Conclusion

This study has presented the effect of initial stress P^* and magnetic field H_0 on surface wave dispersion in bone. The phase velocity and the dimensionless frequency for this problem are obtained from the dimensionless frequency equation. A numerical method has been presented for obtaining the estimates of phase velocity and dimensionless frequencies of vibration of transversely isotropic bone using the half-interval method. The eigenvalues are calculated for different cases and compared with those reported in the absence of initial stress P^* and magnetic field H_0. The effects of initial stress P^* and magnetic field H_0 on the dimensionless frequencies and the phase velocity were indicated by figures.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors, SRM, AT, ATA and KSA contributed to each part of this work equally and read and approved the final version of the manuscript.

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